G Complex Gaussian

G.1 Cartesian formulation

Consider a complex number, $\mathfrak{x} = x_{\mathfrak{R}} + i x_{\mathfrak{I}}$, where $i = \sqrt{-1}$. If we consider the components $x_{\mathfrak{R}}$ and $x_{\mathfrak{I}}$ to be i.i.d. Gaussian RVs, their joint PDF is

$$p\left(x_{\mathfrak{R}}, x_{\mathfrak{I}} \mid \sigma\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{{x_{\mathfrak{R}}}^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{{x_{\mathfrak{I}}}^2}{2\sigma^2}\right) \tag{62}$$

$$p\left(\mathfrak{x} \mid \sigma\right) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|\mathfrak{x}|^2}{2\sigma^2}\right). \tag{63}$$

The expectation of the squared magnitude, rather than the square of the variable, is pertinent[4]:

$$\mathbb{E}\left(|\mathfrak{x}|^2\right) = \mathbb{E}\left({x_{\mathfrak{R}}}^2 + {x_{\mathfrak{I}}}^2\right), \tag{64}$$

$$= \mathbb{E}\left({x_{\mathfrak{R}}}^2\right) + \mathbb{E}\left({x_{\mathfrak{I}}}^2\right), \tag{65}$$

$$= 2\sigma^2. \tag{66}$$

This motivates parameterisation in terms of a variance $\upsilon = 2\sigma^2$:

$$p\left(\mathfrak{x} \mid \upsilon\right) = \frac{1}{\pi\upsilon} \exp\left(-\frac{|\mathfrak{x}|^2}{\upsilon}\right). \tag{67}$$

G.2 Polar formulation

If we make the substitutions

$$a = |\mathfrak{x}| = \sqrt{{x_{\mathfrak{R}}}^2 + {x_{\mathfrak{I}}}^2} \tag{68}$$

$$\theta = \tan^{-1} \frac{x_{\mathfrak{I}}}{x_{\mathfrak{R}}}, \tag{69}$$

so that

$$x_{\mathfrak{R}} = a \cos\theta \tag{70}$$

$$x_{\mathfrak{I}} = a \sin\theta, \tag{71}$$

the Jacobian determinant is

$$J(a, \theta) = \begin{vmatrix} \frac{\partial x_{\mathfrak{R}}}{\partial a} & \frac{\partial x_{\mathfrak{I}}}{\partial a} \\ \frac{\partial x_{\mathfrak{R}}}{\partial \theta} & \frac{\partial x_{\mathfrak{I}}}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -a\sin\theta & a\cos\theta \end{vmatrix} = a. \tag{72}$$

This gives

$$p\left(a, \theta \mid \upsilon\right) = \frac{a}{\pi\upsilon} \exp\left(-\frac{a^2}{\upsilon}\right). \tag{73}$$

Integrating over θ gives the Rayleigh distribution,

$$p\left(a \mid \upsilon\right) = \int_{-\pi}^{\pi} d\theta \, \frac{a}{\pi\upsilon} \exp\left(-\frac{a^2}{\upsilon}\right) \tag{74}$$

$$= \frac{2a}{\upsilon} \exp\left(-\frac{a^2}{\upsilon}\right) \tag{75}$$

Further, if

$$p = |\mathfrak{x}|^2 = a^2, \tag{76}$$

[4] Is there a better source for this?

so

$$a = \sqrt{p} \tag{77}$$

$$\frac{da}{dp} = \frac{1}{2\sqrt{p}} = \frac{1}{2a}, \tag{78}$$

then

$$p\left(p \mid \upsilon\right) = \frac{1}{\upsilon} \exp\left(-\frac{p}{\upsilon}\right). \tag{79}$$

G.3 Summary

The complex Gaussian leads to three common forms dependening on whether one is interested in the distribution of the complex number itself, the magnitude or the squared magnitude:

$$p\left(\mathfrak{x} \mid \upsilon\right) = \frac{1}{\pi\upsilon} \exp\left(-\frac{|\mathfrak{x}|^2}{\upsilon}\right). \tag{80}$$

$$p\left(|\mathfrak{x}| \mid \upsilon\right) = \frac{2|\mathfrak{x}|}{\upsilon} \exp\left(-\frac{|\mathfrak{x}|^2}{\upsilon}\right). \tag{81}$$

$$p\left(|\mathfrak{x}|^2 \mid \upsilon\right) = \frac{1}{\upsilon} \exp\left(-\frac{|\mathfrak{x}|^2}{\upsilon}\right). \tag{82}$$

The first is a function of two variables, the latter two are functions of just one variable, and are Rayleigh and exponential distributions respectively.