G Complex Gaussian

G.1 Cartesian formulation

Consider a complex number, \( x = x_R + i x_I \), where \( i = \sqrt{-1} \). If we consider the components \( x_R \) and \( x_I \) to be i.i.d. Gaussian RVs, their joint PDF is

\[
p(x_R, x_I | \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left( \frac{-x_R^2}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi\sigma}} \exp\left( \frac{-x_I^2}{2\sigma^2} \right)
\]  

(62)

\[
p(|x| | \sigma) = \frac{1}{\pi\sigma} \exp\left( -\frac{|x|^2}{\sigma^2} \right).
\]  

(63)

The expectation of the squared magnitude, rather than the square of the variable, is pertinent:

\[
\mathbb{E}(|x|^2) = \mathbb{E}(x_R^2 + x_I^2),
\]

(64)

\[
= \mathbb{E}(x_R^2) + \mathbb{E}(x_I^2),
\]

(65)

\[
= 2\sigma^2.
\]

(66)

This motivates parameterisation in terms of a variance \( \nu = 2\sigma^2 \):

\[
p(|x| | \nu) = \frac{1}{\pi\nu} \exp\left( -\frac{|x|^2}{\nu} \right).
\]  

(67)

G.2 Polar formulation

If we make the substitutions

\[
a = |x| = \sqrt{x_R^2 + x_I^2}
\]  

(68)

\[
\theta = \tan^{-1} \frac{x_I}{x_R},
\]

(69)

so that

\[
x_R = a \cos \theta
\]

(70)

\[
x_I = a \sin \theta,
\]

(71)

the Jacobian determinant is

\[
J(a, \theta) = \begin{vmatrix} \frac{\partial x_R}{\partial a} & \frac{\partial x_I}{\partial a} \\ \frac{\partial x_R}{\partial \theta} & \frac{\partial x_I}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -a \sin \theta & a \cos \theta \end{vmatrix} = a.
\]

(72)

This gives

\[
p(a, \theta | \nu) = \frac{a}{\pi\nu} \exp\left( -\frac{a^2}{\nu} \right).
\]

(73)

Integrating over \( \theta \) gives the Rayleigh distribution,

\[
p(a | \nu) = \int_{-\pi}^{\pi} d\theta \frac{a}{\pi\nu} \exp\left( -\frac{a^2}{\nu} \right)
\]

(74)

\[
= \frac{2a}{\nu} \exp\left( -\frac{a^2}{\nu} \right)
\]

(75)

Further, if

\[
p = |x|^2 = a^2,
\]

(76)

\[\text{Is there a better source for this?}\]
so

\[ a = \sqrt{p} \quad (77) \]

\[ \frac{da}{dp} = \frac{1}{2\sqrt{p}} = \frac{1}{2a}, \quad (78) \]

then

\[ p (p | \nu) = \frac{1}{\nu} \exp \left( -\frac{p}{\nu} \right). \quad (79) \]

**G.3 Summary**

The complex Gaussian leads to three common forms depending on whether one is interested in the distribution of the complex number itself, the magnitude or the squared magnitude:

\[ p (x | \nu) = \frac{1}{\pi \nu} \exp \left( -\frac{|x|^2}{\nu} \right). \quad (80) \]

\[ p (|x| | \nu) = \frac{2|x|}{\nu} \exp \left( -\frac{|x|^2}{\nu} \right). \quad (81) \]

\[ p (|x|^2 | \nu) = \frac{1}{\nu} \exp \left( -\frac{|x|^2}{\nu} \right). \quad (82) \]

The first is a function of two variables, the latter two are functions of just one variable, and are Rayleigh and exponential distributions respectively.