E Damped system

E.1 Derivation

This is mostly from Wikipedia[3]. Take a simple system with a mass, a spring and a damper (figure 1): There are

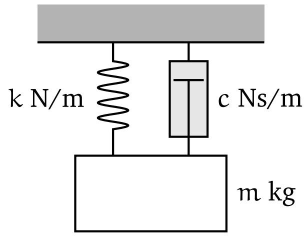

Figure 1: Mass spring damper system.

three forces:

1. A force from the spring, proportional to displacement
$$f_s = -ky. \tag{162}$$
2. A force from the damper, proportional to velocity
$$f_d = -c\frac{dy}{dt}. \tag{163}$$
3. An (optional) external input force
$$f_i. \tag{164}$$

Putting these into Newton's second law,

$$\text{force} = \text{mass} \times \text{acceleration} \tag{165}$$
$$f_i - ky - c\frac{dy}{dt} = m\frac{d^2y}{dt^2} \tag{166}$$
$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f_i \tag{167}$$

or, writing y as $y(t)$ and f_i and $f(t)$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t). \tag{168}$$

The Laplace transform[4] of this is

$$m\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + c\left[sY(s) - y(0)\right] + kY(s) = F(s) \tag{169}$$
$$Y(s)(ms^2 + cs + k) = msy(0) + m\dot{y}(0) + cy(0) + F(s) \tag{170}$$
$$Y(s) = \frac{msy(0) + m\dot{y}(0) + cy(0) + F(s)}{ms^2 + cs + k}. \tag{171}$$

So, there is a single zero and a double pole.

E.2 Reparameterisation

The two poles are at

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}, \tag{172}$$

[3] https://en.wikipedia.org/wiki/Damping
[4] https://en.wikipedia.org/wiki/Laplace_transform

and are coincident when $c^2 = 4mk$. Further, when $c = 0$ the poles are pure imaginary at

$$s = \pm j\sqrt{\frac{k}{m}}, \tag{173}$$

at which the system will resonate. Hence, define

$$\zeta^2 \triangleq \frac{c^2}{4mk} \implies c = 2\zeta\sqrt{mk}, \tag{174}$$

and

$$\omega_0^2 \triangleq \frac{k}{m} \implies k = m\omega_0^2. \tag{175}$$

In this case, the system becomes

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y(t) = f(t)/m. \tag{176}$$

Note also that we can define

$$x(t) \triangleq \frac{f(t)}{k} \implies f(t) = kx(t), \tag{177}$$

which has the effect of parameterising the force as a distance $x(t)$ being the distance that the spring would move if subjected to $f(t)$. We then have

$$\frac{f(t)}{m} = \frac{k}{m}x(t) = \omega_0^2 x(t) \tag{178}$$

so

$$Y(s) = \frac{sy(0) + \dot{y}(0) + 2\zeta\omega_0 y(0) + \omega_0^2 X(s)}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \tag{179}$$

with poles

$$s_1 = -\omega_0\zeta + \omega_0\sqrt{\zeta^2 - 1}, \tag{180}$$
$$s_2 = -\omega_0\zeta - \omega_0\sqrt{\zeta^2 - 1}, \tag{181}$$

or, if $\zeta = 1$ (critical damping)

$$s_1 = s_2 = -\omega_0. \tag{182}$$

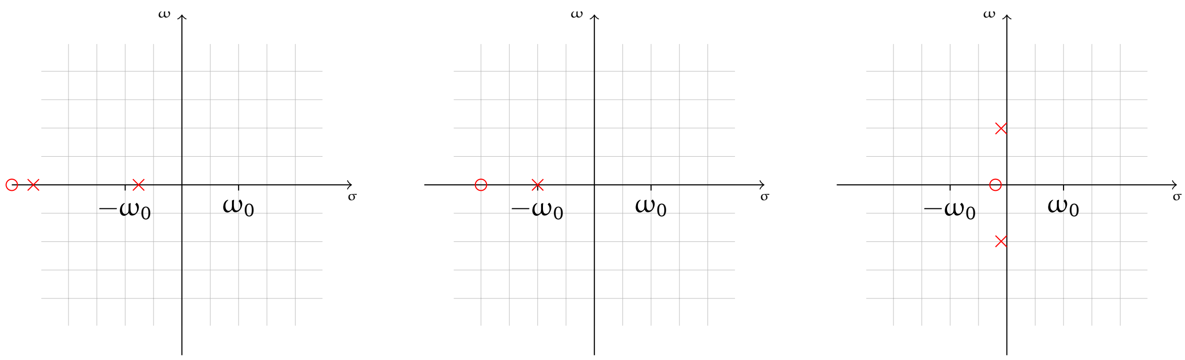

Figure 2: Damped systems for $\zeta = 1.5, 1$ and 0.1 respectively, and assuming $\dot{y}(0) = 0$ and $X(s) = 0$.

In general, for the system to do something interesting, it either needs non-zero initial conditions, or some driving force. These are discussed below.

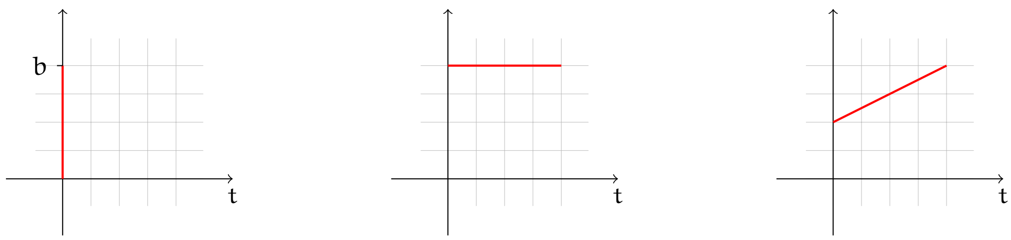

Figure 3: Driven systems: Impulse, step and linear respectively.

E.3 General solutions

E.3.1 General case

We have

$$Y(s) = \frac{sy(0) + \dot{y}(0) + 2\zeta\omega_0 y(0) + \omega_0^2 X(s)}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{183}$$

with

$$s_1 = -\omega_0\zeta + \omega_0\sqrt{\zeta^2 - 1}, \tag{184}$$
$$s_2 = -\omega_0\zeta - \omega_0\sqrt{\zeta^2 - 1}. \tag{185}$$

The solution does not have a particularly attractive form:

$$y(t) = \frac{y(0)}{-s_2 + s_1}\left(s_1 e^{s_1 t} - s_2 e^{s_2 t}\right) + \frac{\dot{y}(0) + 2\zeta\omega_0 y(0) + \omega_0^2 X(s)}{-s_2 + s_1}\left(e^{s_1 t} - e^{s_2 t}\right) \tag{186}$$
$$= e^{-\zeta\omega_0 t}\left[A e^{\sqrt{\zeta^2-1}\omega_0 t} - B e^{-\sqrt{\zeta^2-1}\omega_0 t}\right] \tag{187}$$

where

$$A = \frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_0 y(0) + \dot{y}(0) + \omega_0^2 X(s)}{2\omega_0\sqrt{\zeta^2 - 1}} \tag{188}$$
$$B = \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_0 y(0) + \dot{y}(0) + \omega_0^2 X(s)}{2\omega_0\sqrt{\zeta^2 - 1}}. \tag{189}$$

The natural response is obtained for $X(s) = 0$; the form does not simplify.

E.3.2 Impulse driven

If we assume steady state initial conditions and an impulsive driving force, $X(s) = 1$, we have

$$Y(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{190}$$
$$y(t) = \frac{\omega_0}{2\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t}\left(e^{\sqrt{\zeta^2-1}\omega_0 t} - e^{-\sqrt{\zeta^2-1}\omega_0 t}\right) \tag{191}$$

and if $\zeta^2 < 1$,

$$= \frac{\omega_0}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0 t}\sin\left(\sqrt{1 - \zeta^2}\omega_0 t\right). \tag{192}$$

i.e., an exponentially weighted sinusoid.

E.4 Critically damped solutions

E.4.1 Impulse driven

Assuming critical damping and an impulse driving force,

$$\begin{aligned} Y(s) &= \frac{sy(0)}{(s+\omega_0)^2} + \frac{\dot{y}(0) + 2\omega_0 y(0) + \omega_0^2}{(s+\omega_0)^2} && (193) \\ y(t) &= y(0)(1-\omega_0 t)e^{-\omega_0 t}\,u(t) + [\dot{y}(0) + 2\omega_0 y(0) + \omega_0^2]te^{-\omega_0 t}\,u(t), && (194) \\ &= (A + Bt)e^{-\omega_0 t}\,u(t), && (195) \end{aligned}$$

where $u(t)$ is the Heaviside step function and

$$\begin{aligned} A &= y(0), && (196) \\ B &= \dot{y}(0) + \omega_0 y(0) + \omega_0^2. && (197) \end{aligned}$$

For steady state initial conditions,

$$y(t) = \omega_0^2 te^{-\omega_0 t}\,u(t). \qquad (198)$$

E.4.2 Step driven

Assuming critical damping and a step driving force,

$$\begin{aligned} x(t) &= b && (199) \\ X(S) &= \frac{b}{s} && (200) \\ Y(s) &= \frac{sy(0)}{(s+\omega_0)^2} + \frac{\dot{y}(0) + 2\omega_0 y(0)}{(s+\omega_0)^2} + \frac{b\omega_0^2}{s(s+\omega_0)^2} && (201) \\ y(t) &= y(0)(1-\omega_0 t)e^{-\omega_0 t}\,u(t) + [\dot{y}(0) + 2\omega_0 y(0)]te^{-\omega_0 t}\,u(t) + b - b(1+\omega_0 t)e^{-\omega_0 t}\,u(t), && (202) \\ &= \left[b - (A + Bt)e^{-\omega_0 t}\right]u(t), && (203) \end{aligned}$$

$$\begin{aligned} A &= y(0) + b, && (204) \\ B &= \dot{y}(0) + \omega_0(y(0) + b). && (205) \end{aligned}$$

For steady state initial conditions and $b = 1$,

$$y(t) = \left[1 - (1+\omega_0 t)e^{-\omega_0 t}\right]u(t). \qquad (206)$$

E.4.3 Linear function driven

In this case,

$$\begin{aligned} x(t) &= b + mt && (207) \\ X(s) &= \frac{b}{s} + \frac{m}{s^2}, && (208) \end{aligned}$$

so it is the step case with extra terms

$$\frac{m\omega_0^2}{s^2(s+\omega_0)^2} \qquad (209)$$

and

$$-\frac{m}{\omega_0}\left(2 - \omega_0 t - (\omega_0 t + 2)e^{-\omega_0 t}\right) \qquad (210)$$

so

$$y(t) = \left[-\frac{2m}{\omega_0} + mt + b - (A + Bt)e^{-\omega_0 t}\right]u(t), \qquad (211)$$

with

$$\begin{aligned} A &= y(0) + b + \frac{2m}{\omega_0}, && (212) \\ B &= \dot{y}(0) + \omega_0(y(0) + b) + m. && (213) \end{aligned}$$

E.5 Discretisation

E.5.1 Impulse driven, general case

First sample the time domain with $t = nT$,

$$y(t) = \frac{\omega_0}{\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \sin\left(\sqrt{\zeta^2 - 1}\omega_0 t\right) \tag{214}$$

$$y(nT) = \frac{\omega_0}{\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 nT} \sin\left(\sqrt{\zeta^2 - 1}\omega_0 nT\right) \tag{215}$$

$$= \frac{\theta_0^2}{\theta T} r^n \sin(n\theta) \tag{216}$$

where

$$\theta_0 = \omega_0 T, \tag{217}$$

$$\theta = \theta_0 \sqrt{\zeta^2 - 1}, \tag{218}$$

$$r = e^{-\zeta\theta_0}, \tag{219}$$

Now transform

$$Y(z) = \frac{\theta_0^2}{\theta T} \frac{rz^{-1} \sin(\theta)}{1 - 2rz^{-1} \cos(\theta) + r^2 z^{-2}} \tag{220}$$

$$= \frac{\theta_0^2}{\theta T} \frac{rz \sin(\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}. \tag{221}$$

E.5.2 Impulse driven, critically damped

Sampling is the same as above:

$$y(t) = \omega_0^2 t e^{-\omega_0 t} u(t) \tag{222}$$

$$y(nT) = \omega_0^2 nT e^{-\omega_0 nT} u(0) \tag{223}$$

$$= \frac{\theta_0^2}{T} n r^n u(0). \tag{224}$$

Then the transform is also tabulated

$$Y(z) = \frac{\theta_0^2}{T} \frac{rz^{-1}}{(1 - rz^{-1})^2} \tag{225}$$

$$= \frac{\theta_0^2}{T} \frac{rz}{(z - r)^2}. \tag{226}$$

E.5.3 Critically damped natural response from offset

This like the impulse driven case, but with non-zero $y(0)$ and no input:

$$y(t) = (y(0) + \omega_0 y(0) t)\, e^{-\omega_0 t} u(t) \tag{227}$$

$$y(nT) = (y(0) + \omega_0 y(0) nT)\, e^{-\omega_0 nT} u(0) \tag{228}$$

$$= (y(0) r^n + \theta_0 y(0) n r^n)\, u(0) \tag{229}$$

$$Y(z) = y(0) \frac{1}{1 - rz^{-1}} + \theta_0 y(0) \frac{rz^{-1}}{(1 - rz^{-1})^2} \tag{230}$$

$$= y(0) \frac{(1 + r\theta_0) z - r}{(z - r)^2} \tag{231}$$

E.5.4 Basic exponential decay

This is trivial, but for reference:

$$y(t) = e^{-\omega_0 t} \tag{232}$$

$$y(nT) = e^{-\omega_0 nT} \tag{233}$$

$$= r^n \tag{234}$$

$$Y(z) = \frac{1}{1 - rz^{-1}} \tag{235}$$

$$= \frac{z}{z - r} \tag{236}$$