A Fourier analysis

A.1 Fourier series

A.1.1 Real

Start off with the identity[1]

$$a\cos(x) + b\sin(x) = \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}\left(\frac{b}{a}\right)\right) \tag{1}$$

which says that you can represent a sinusoid with arbitrary magnitude and phase as a sum of a sine and a cosine. Consider summing harmonics of this sinusoid:

$$f(x) = \sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx). \tag{2}$$

Notice that $n = 0$ just leaves $f(x) = a_0$, which is a constant offset.

Now consider the integral

$$I = \int_{-\pi}^{\pi} \cos(\theta)\cos(\phi)\, d\theta. \tag{3}$$

It's basically zero for everything except $\phi = \theta$, in which case

$$I = \int_{-\pi}^{\pi} \cos^2(\theta)\, d\theta = \pi. \tag{4}$$

So, we can solve for a_n and b_n by multiplying by a sinusoid and integrating:

$$\int_{-\pi}^{\pi} f(x)\cos(kx)\, dx = \int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx)\right)\cos(kx)\, dx \tag{5}$$

$$= \pi a_k \tag{6}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\, dx \tag{7}$$

and similarly

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx. \tag{8}$$

Equations 7 and 8 define the Fourier series.

A.1.2 Complex

Now bring in a couple of identities

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{9}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}, \tag{10}$$

[1] https://en.wikipedia.org/wiki/Trig_identities

where $j = \sqrt{-1}$ and $1/j = -j$. This allows us to write

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{11}$$

$$= \sum_{n=0}^{\infty} a_n \frac{e^{jnx} + e^{-jnx}}{2} + b_n \frac{e^{jnx} - e^{-jnx}}{2j} \tag{12}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (a_n - jb_n)e^{jnx} + (a_n + jb_n)e^{-jnx} \tag{13}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (a_n - jb_n)e^{jnx} + \frac{1}{2} \sum_{n=-\infty}^{-1} (a_n + jb_n)e^{jnx} \tag{14}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{jnx} \tag{15}$$

where

$$c_n = \frac{1}{2}(a_n - jb_n) \tag{16}$$

$$a_{-n} = a_n \tag{17}$$

$$b_{-n} = -b_n \tag{18}$$

and

$$b_0 = 0. \tag{19}$$

Notice that it's double sided; a single sinusoid is now split across positive and negative components.

The inverse (actually forward) case follows similarly:

$$c_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos(nx) - j\sin(nx))\, dx \tag{20}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\frac{e^{jnx} + e^{-jnx}}{2} - j\frac{e^{jnx} - e^{-jnx}}{2j} \right) dx \tag{21}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-jnx}\, dx. \tag{22}$$

Equations 22 and 15 define respectively the complex forward and inverse Fourier series.

A.1.3 Reconstruction

It's tempting to say that the real coefficients are obvious from the complex ones; in practice it's necessary to reconstruct them formally:

$$a_0 = 2c_0 \tag{23}$$

$$a_n = c_n + c_{-n} \tag{24}$$

$$b_n = j(c_n - c_{-n}) \tag{25}$$

A.2 Fourier transform

A.2.1 Cycles

So far, x is measured in radians. Instead, specify it in, say, seconds. This amounts to a substitution

$$t = \frac{Tx}{2\pi} \tag{26}$$

$$x = \frac{2\pi t}{T} \tag{27}$$

$$dx = \frac{2\pi}{T}\, dt \tag{28}$$

so

$$x = \pi \implies t = \frac{T}{2} \tag{29}$$
$$x = -\pi \implies t = -\frac{T}{2} \tag{30}$$

The transform is then

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi j t n/T}\,dt. \tag{31}$$

and the inverse is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi j t n/T} \tag{32}$$

One can think of n as measuring cycles, so $f = n/T$ is cycles per second, or Hertz. Define

$$f = \lim_{T\to\infty} \frac{n}{T} \tag{33}$$
$$\Delta f = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T} \tag{34}$$

so the inverse transform becomes a Riemann sum

$$f(t) = \sum_{n=-\infty}^{\infty} T c_n e^{2\pi j t n/T} \Delta f. \tag{35}$$

Further, if the complex c_n measures, say, energy, then the continuous function measures energy density. So,

$$F(f) = \lim_{T\to\infty} \frac{c_n}{\Delta f} = \lim_{T\to\infty} T c_n. \tag{36}$$

Consolidating the above reasoning and letting $\Delta f \to df$,

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-2\pi j t f}\,dt \tag{37}$$
$$f(t) = \int_{-\infty}^{\infty} F(f)e^{2\pi j t f}\,df. \tag{38}$$

Equations 37 and 38 define respectively the Fourier and inverse Fourier transforms.

A.2.2 Angular frequency

It is possible to change variable from f in Hz (cycles per second) to ω in radians per second. So, if

$$\omega = 2\pi f \tag{39}$$
$$f = \frac{\omega}{2\pi} \tag{40}$$
$$df = \frac{1}{2\pi}\,d\omega \tag{41}$$

gives

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{42}$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega. \tag{43}$$

This is the one that makes sense in the context of Laplace and z-transforms.

A.3 Discrete time Fourier transform

In the context of sampling, time becomes discrete. If the sampling frequency is F_s,

$$T = \frac{1}{F_s} \tag{44}$$
$$t = nT \tag{45}$$
$$x_n = Tf(nT) \tag{46}$$

so we can write

$$f(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - nT), \tag{47}$$

and substituting,

$$F(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega nT}\, dt \tag{48}$$
$$x_n = \frac{T}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega nT}\, d\omega. \tag{49}$$

Now redefine ω to lie in the range $-\pi$ to π

$$\omega' = 2\pi \frac{f}{F_s} = 2\pi f T = \omega T \tag{50}$$
$$\omega = \frac{\omega'}{T} \tag{51}$$
$$d\omega = \frac{1}{T}\, d\omega' \tag{52}$$

This leads to:

$$F(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n} \tag{53}$$
$$x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n}\, d\omega. \tag{54}$$

Equations 53 and 54 define respectively the Discrete Time Fourier and inverse Discrete Time Fourier transforms.

A.4 Discrete Fourier transform

A.4.1 From Fourier series

Recall the complex Fourier series, but with k replacing n for the discrete frequency points:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-jkx}\, dx \tag{55}$$
$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{jkx} \tag{56}$$

It is already discrete in frequency; to make it discrete in time too, make a similar substitution as before. First change the period of x (2π) into something that can be discretised:

$$t = \frac{N}{2\pi} x \tag{57}$$
$$x = \frac{2\pi}{N} t \tag{58}$$
$$dx = \frac{2\pi}{N}\, dt \tag{59}$$
$$x = -\pi \implies t = -\frac{N}{2} \tag{60}$$
$$x = \pi \implies t = \frac{N}{2} \tag{61}$$

Substituting,

$$c_k = \frac{1}{N}\int_{-N/2}^{N/2} f(t)e^{-j2\pi kt/N}\,dt \tag{62}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/N} \tag{63}$$

Now discretise; note that c_k is now periodic so the range changes

$$t = n \tag{64}$$

$$f(t) = \sum_{n=-N/2+1}^{N/2} f(t)\delta(t-n) \tag{65}$$

$$= \sum_{n=-N/2+1}^{N/2} x_n\delta(t-n) \tag{66}$$

so,

$$c_k = \frac{1}{N}\sum_{n=-N/2+1}^{N/2} x_n e^{-j2\pi kn/N} \tag{67}$$

$$x_n = \sum_{k=-N/2+1}^{N/2} c_k e^{j2\pi kn/N} \tag{68}$$

A.4.2 From discrete time Fourier transform

The DTFT is already discrete in time; now make it discrete in frequency.

$$f = \frac{N}{2\pi}\omega \tag{69}$$

$$\omega = \frac{2\pi}{N}f \tag{70}$$

$$d\omega = \frac{2\pi}{N}\,df \tag{71}$$

$$\omega = -\pi \implies f = -\frac{N}{2} \tag{72}$$

$$\omega = \pi \implies f = \frac{N}{2} \tag{73}$$

Substituting,

$$F(f) = \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi fn/N} \tag{74}$$

$$x_n = \frac{1}{N}\int_{-N/2}^{N/2} F(f)e^{j2\pi fn/N}\,df. \tag{75}$$

Now discretise; note that x_n is now periodic so the range changes

$$f = k \tag{76}$$

$$F(f) = \sum_{n=-N/2+1}^{N/2} F(f)\delta(f-k) \tag{77}$$

$$= \sum_{n=-N/2+1}^{N/2} \mathfrak{f}_k\delta(t-n) \tag{78}$$

so,

$$f_k = \sum_{n=-N/2+1}^{N/2} x_n e^{-j2\pi kn/N} \tag{79}$$

$$x_n = \frac{1}{N} \sum_{k=-N/2+1}^{N/2} f_k e^{j2\pi kn/N} \tag{80}$$

i.e., the normaliser is on the inverse transform instead of the forward transform as it would be for the series based derivation.

A.4.3 Convention

The discrete Fourier transform pair is normally defined as

$$f_k = \sum_{k=0}^{N-1} x_n e^{-j2\pi kn/N} \tag{81}$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{j2\pi kn/N}, \tag{82}$$

i.e., the one based on DTFT. It's time shifted too, in the sense of matlab or numpy's `fftshift()`.