P Gaussian

P.1 Marginalisation over variance

Start with a Gaussian with zero mean and variance $\nu$.

$$p(x | \nu) = \frac{1}{\sqrt{2\pi\nu}} \exp\left(-\frac{x^2}{2\nu}\right). \quad (270)$$

Now marginalise with an inverse-gamma prior on the variance:

$$p(x) = \int_0^\infty d\nu p(x | \nu) p(\nu) \quad (271)$$

$$= \int_0^\infty d\nu \frac{1}{\sqrt{2\pi\nu}} \exp\left(-\frac{x^2}{2\nu}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \nu^{-\alpha-1} \exp\left(-\frac{\beta}{\nu}\right) \quad (272)$$

$$= \frac{1}{\sqrt{2\pi} \Gamma(\alpha)} \int_0^\infty d\nu \nu^{-(\alpha+1/2)-1} \exp\left(-\frac{x^2/2 + \beta}{\nu}\right). \quad (273)$$

That integral is the normaliser for an inverse-gamma distribution, so

$$p(x) = \frac{1}{\sqrt{2\pi} \Gamma(\alpha)} \left(\frac{x^2}{2} + \frac{\beta}{\nu}\right)^{-\alpha-1/2} \Gamma(\alpha+1/2) \quad (274)$$

$$= \frac{\beta^\alpha}{\sqrt{2\pi} \Gamma(1/2)\Gamma(\alpha)} \left(\frac{x^2}{2} + \frac{\beta}{\nu}\right)^{-\alpha-1/2} \quad (275)$$

$$= \frac{1}{\sqrt{2\pi} B(1/2, \alpha)} \left(\frac{x^2}{2\beta} + 1\right)^{-\alpha-1/2} \quad (276)$$

This is a Student’s t distribution if

$$\alpha = \nu/2 \quad (277)$$
$$2\beta = \nu \quad (278)$$

so

$$\alpha = \beta = \nu/2 \quad (279)$$
Figure 1: Example marginalised distributions.