P Gaussian

P.1 Marginalisation over variance

Start with a Gaussian with zero mean and variance υ.

$$p(x \mid \upsilon) = \frac{1}{\sqrt{2\pi\upsilon}} \exp\left(-\frac{x^2}{2\upsilon}\right). \tag{270}$$

Now marginalise with an inverse-gamma prior on the variance:

$$\begin{aligned} p(x) &= \int_0^\infty d\upsilon\, p(x \mid \upsilon)\, p(\upsilon) && (271)\\ &= \int_0^\infty d\upsilon\, \frac{1}{\sqrt{2\pi\upsilon}} \exp\left(-\frac{x^2}{2\upsilon}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \upsilon^{-\alpha-1} \exp\left(-\frac{\beta}{\upsilon}\right) && (272)\\ &= \frac{1}{\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty d\upsilon\, \upsilon^{-(\alpha+1/2)-1} \exp\left(-\frac{x^2/2+\beta}{\upsilon}\right). && (273) \end{aligned}$$

That integral is the normaliser for an inverse-gamma distribution, so

$$\begin{aligned} p(x) &= \frac{1}{\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1/2)}{(x^2/2+\beta)^{\alpha+1/2}} && (274)\\ &= \frac{\beta^\alpha}{\sqrt{2}} \frac{\Gamma(\alpha+1/2)}{\Gamma(1/2)\Gamma(\alpha)} \left(\frac{x^2}{2}+\beta\right)^{-\alpha-1/2} && (275)\\ &= \frac{1}{\sqrt{2\beta}} \frac{1}{B(1/2,\alpha)} \left(\frac{x^2}{2\beta}+1\right)^{-\alpha-1/2}. && (276) \end{aligned}$$

This is a Student's t distribution if

$$\begin{aligned} \alpha &= \nu/2 && (277)\\ 2\beta &= \nu && (278) \end{aligned}$$

so

$$\alpha = \beta = \nu/2 \tag{279}$$

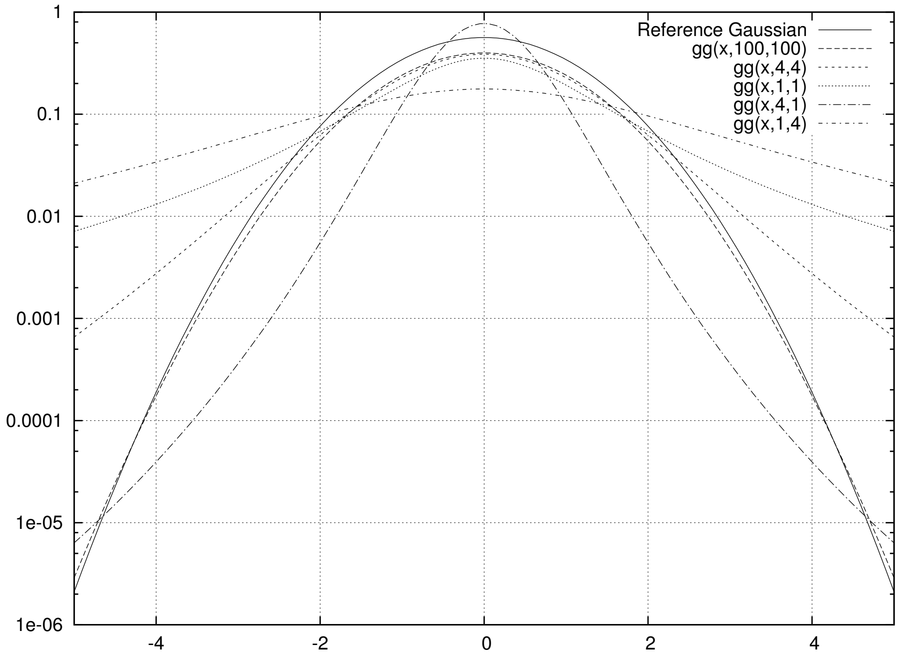

Figure 1: Example marginalised distributions.