C Laplace transform

C.1 Continuous

Start with the Fourier transform pair in terms of angular frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{97}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega. \tag{98}$$

The forward transform is known not to converge for certain signals. However, it can be forced to converge by multiplying by an exponential:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}e^{-\sigma t}\,dt \tag{99}$$

where σ is some arbitrary constant. If we write

$$s = \sigma + j\omega, \tag{100}$$

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt. \tag{101}$$

Equation 101 is the bilateral Laplace transform. The inverse is almost available from the substitution:

$$\omega = \frac{s-\sigma}{j} \tag{102}$$

$$d\omega = \frac{1}{j}\,ds \tag{103}$$

$$\omega = \infty \implies s = \sigma + j\infty \tag{104}$$

$$\omega = -\infty \implies s = \sigma - j\infty \tag{105}$$

giving

$$f(t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} F(s)e^{st}\,ds \tag{106}$$

where γ defines a region where the integral converges.

Note:

- That inverse is not a rigorous derivation.
- The Laplace transform is usually the unilateral one, i.e.,

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}e^{-\sigma t}\,dt. \tag{107}$$

C.2 Discrete

To derive the discrete time version, proceed in the same way as was done for the discrete time Fourier transform:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \tag{108}$$

$$= \sum_{n=-\infty}^{\infty} x_n\delta(t-nT), \tag{109}$$

which is a sampled form of $x(t)$. Now substitute into the (bilateral) Laplace transform:

$$F(s) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_n\delta(t-nT)e^{-st}\,dt, \tag{110}$$

$$= \sum_{n=-\infty}^{\infty} x_n\int_{-\infty}^{\infty}\delta(t-nT)e^{-st}\,dt, \tag{111}$$

$$= \sum_{n=-\infty}^{\infty} x_n e^{-snT}. \tag{112}$$

then if

$$z = e^{sT}, \tag{113}$$

we have

$$F(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}. \tag{114}$$

The inverse is

$$x_n = \frac{1}{2\pi j} \oint_C F(z) z^{n-1}\, dz. \tag{115}$$

Equations 114 and 115 define the forward and inverse z-transform.

C.3 Laurent series

The above is not especially rigorous. In fact, Laplace himself was playing with something closer to the z-transform. A more formal origin of this is the Laurent series[2]. What Laurent probably did was start with this generalisation of the Cauchy integral formula

$$x_n = \frac{1}{2\pi j} \oint_\gamma \frac{F(z)}{(z-c)^{n+1}}\, dz, \tag{116}$$

and show that it evaluates to

$$F(z) = \sum_{n=-\infty}^{\infty} x_n (z-c)^n, \tag{117}$$

which is the Laurent series. It's a complex, double sided version of the Taylor series that transforms between a discrete real sequence and a continuous complex one. Setting $c = 0$ (cf. the Maclaurin series) and flipping the sign of n gives the z-transform.

C.4 Laplace transform from z-transform

Beginning with the z-transform pair, write

$$s = \log z \tag{118}$$

$$z = e^s \tag{119}$$

$$dz = e^s\, ds \tag{120}$$

so that

$$F(s) = \sum_{n=-\infty}^{\infty} x_n e^{-sn} \tag{121}$$

$$x_n = \frac{1}{2\pi j} \int_C F(s) e^{sn}\, ds, \tag{122}$$

and the contour C is now a line integral from $-j \log \pi$ to $j \log \pi$. Now follow the procedure for deriving the Fourier transform from the Fourier series. If the period between samples is T, we have

$$t = \lim_{T \to 0} nT \tag{123}$$

$$\Delta t = (n+1)T - nT = T \tag{124}$$

$$n = \frac{t}{T} \tag{125}$$

$$x_n = Tf(nT) \tag{126}$$

[2] https://en.wikipedia.org/wiki/Laurent_series

To get rid of the T in the exponential, define

$$s' = \frac{s}{T} \tag{127}$$
$$s = s'T \tag{128}$$
$$ds = T\,ds' \tag{129}$$

so the transform pair becomes

$$F(s') = \sum_{n=-\infty}^{\infty} Tf(nT)e^{-s't}\frac{1}{T}\Delta t \tag{130}$$
$$Tf(nT) = \frac{1}{2\pi j}\int_C F(s')e^{s't}T\,ds'. \tag{131}$$

Note that all the lone T terms cancel. Letting $T \to 0$, and replacing s' with s,

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \tag{132}$$
$$f(t) = \frac{1}{2\pi j}\int_C F(s)e^{st}\,ds. \tag{133}$$

As for the line integral, $s' = s/T$, so it is now from from $-j\infty$ to $j\infty$. This is the Laplace transform pair.

In the above, I haven't defined t to be time. However, by comparison with the Fourier transform, it's clear that it can be interpretted as time.