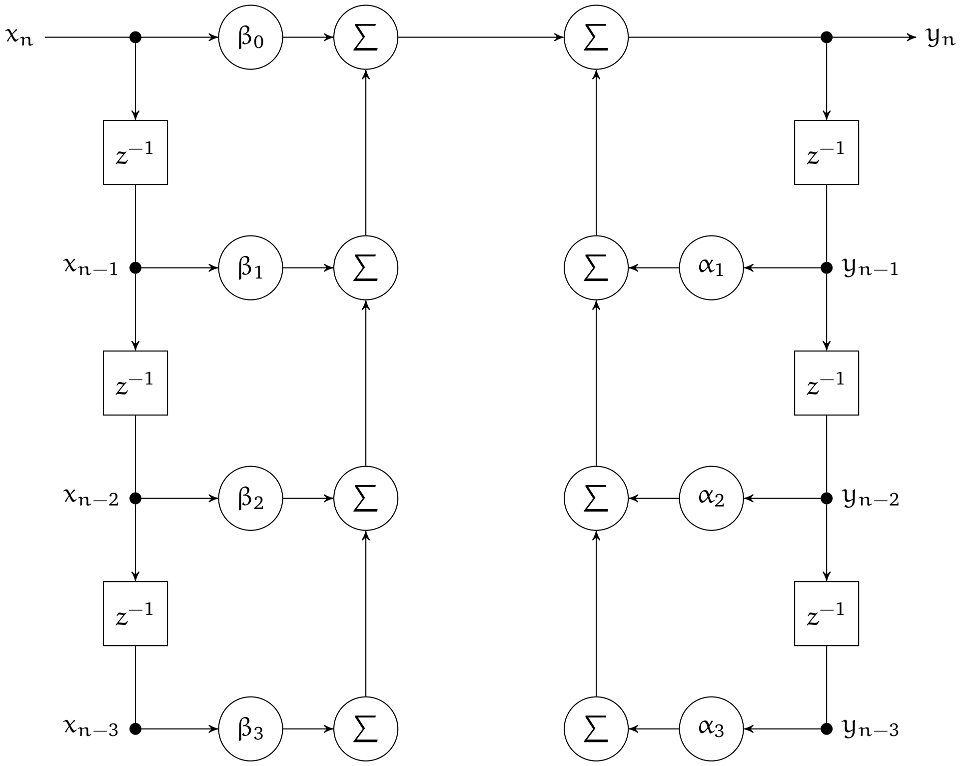
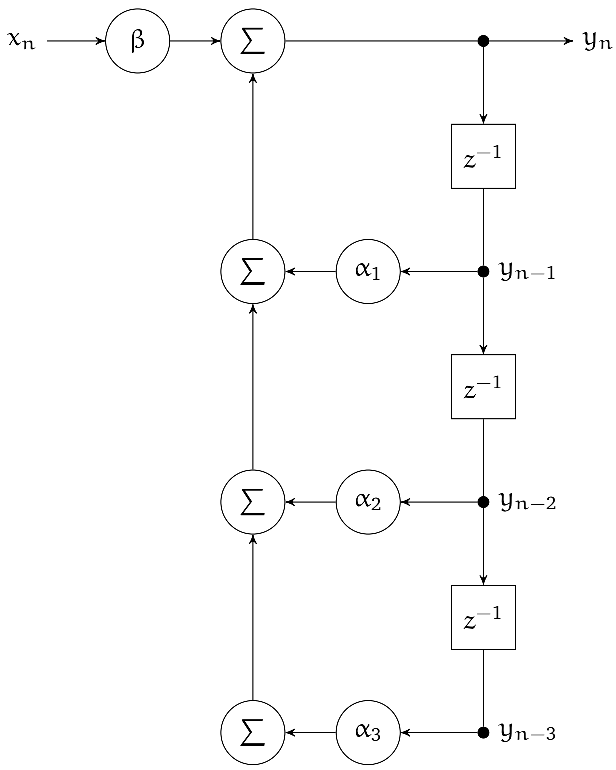
A Linear Model

A.1 Generative model

This is a general filter:



All pole is that with zeros removed:



Notice that we still have β_0, a gain term, written here as β.

A.2 Formulation

Say we have:

- N acoustic observations, $\boldsymbol{y} = (y_{t-N+1}, y_{t-N+2}, \ldots, y_t)^\mathsf{T}$. They tend to be time indexed, in which case the most recent is time t.
- A vector of P linear prediction coefficients, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_P)^\mathsf{T}$.

Define a model where the observation is a function of a current input, or *excitation*, and previous *observations* (outputs):

$$y_n = \beta x_n + \sum_{p=1}^{P} \alpha_p y_{n-p}. \tag{1}$$

It's more easily expressed in matrix form. For a window size N:

$$\underbrace{\begin{pmatrix} y_{t-N+P+1} \\ y_{t-N+P+2} \\ \vdots \\ y_t \end{pmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{pmatrix} y_{t-N+1} & y_{t-N+2} & \cdots & y_{t-N+P} \\ y_{t-N+2} & y_{t-N+3} & \cdots & y_{t-N+P+1} \\ \vdots & \vdots & & \vdots \\ y_{t-P} & y_{t-P+1} & \cdots & y_{t-1} \end{pmatrix}}_{\mathbf{Y} \text{ (Tall and thin, overdetermined)}} \underbrace{\begin{pmatrix} \alpha_P \\ \alpha_{P-1} \\ \vdots \\ \alpha_1 \end{pmatrix}}_{\alpha} + \beta \boldsymbol{x} \tag{2}$$

Notice that there are $N - P$ equations; we ignore the ones for which the previous outputs are not available. Ideally, $N \gg P$, so there are lots of samples.

A.3 Estimation of parameters

Say the excitation is a Gaussian with zero mean and unit variance:

$$p(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}^{N-P}} \exp\left(-\frac{1}{2}\boldsymbol{x}^\mathsf{T}\boldsymbol{x}\right). \tag{3}$$

Make a change of variable

$$\boldsymbol{y} = \mathbf{Y}\boldsymbol{\alpha} + \beta\boldsymbol{x} \tag{4}$$

$$\boldsymbol{x} = \frac{1}{\beta}(\boldsymbol{y} - \mathbf{Y}\boldsymbol{\alpha}) \tag{5}$$

The Jacobian is $1/\beta$ for each of the $N - P$ equations, so after substitution, we get

$$p(\boldsymbol{y} \mid \boldsymbol{\alpha}) = \frac{1}{\sqrt{2\pi\beta^2}^{N-P}} \exp\left(-\frac{1}{2\beta^2}(\boldsymbol{y} - \mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\boldsymbol{y} - \mathbf{Y}\boldsymbol{\alpha})\right). \tag{6}$$

Finally, differentiate w.r.t. $\boldsymbol{\alpha}$ and equate to zero[1]:

$$\hat{\boldsymbol{\alpha}} = (\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\boldsymbol{y}. \tag{7}$$

This is not specific to LPC; it's a standard statistical result.

This bit *is* specific to LPC:

- $\mathbf{Y}^\mathsf{T}\mathbf{Y}$ is *basically* the *autocorrelation*, with a few edge effects.
- $\mathbf{Y}^\mathsf{T}\boldsymbol{y}$ is *also* basically the autocorrelation, with a few edge effects.

$$\hat{\boldsymbol{\alpha}} = \begin{pmatrix} r_0 & r_1 & \ldots & r_{P-1} \\ r_1 & r_0 & \ldots & r_{P-2} \\ \vdots & \vdots & & \vdots \\ r_{P-1} & r_{P-2} & \ldots & r_0 \end{pmatrix}^{-1} \begin{pmatrix} r_P \\ r_{P-1} \\ \vdots \\ r_1 \end{pmatrix} \tag{8}$$

[1] You can sort of see the result is going to be a rearrangement of $\boldsymbol{y} = \mathbf{Y}\alpha$

There is a $1/(N-P)$ term that cancels in the above; $\mathbf{Y}^\mathsf{T}\mathbf{Y}$ and $\mathbf{Y}^\mathsf{T}\mathbf{y}$ are not normalised, but the autocorrelation terms are.

The gain is just the variance:

$$\begin{aligned}
\hat{\beta}^2 &= \frac{1}{N-P}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha}) && (9)\\
&= \frac{1}{N-P}(\mathbf{y}-\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y})^\mathsf{T}(\mathbf{y}-\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y}) && (10)\\
&= \frac{1}{N-P}\left(\mathbf{y}^\mathsf{T}\mathbf{y}+\mathbf{y}^\mathsf{T}\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1^\mathsf{T}}\mathbf{Y}^\mathsf{T}\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y}-2\mathbf{y}^\mathsf{T}\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y}\right) && (11)\\
&= \frac{1}{N-P}\left(\mathbf{y}^\mathsf{T}\mathbf{y}-\mathbf{y}^\mathsf{T}\mathbf{Y}(\mathbf{Y}^\mathsf{T}\mathbf{Y})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y}\right) && (12)\\
&= \frac{1}{N-P}\left(\mathbf{y}^\mathsf{T}\mathbf{y}-\boldsymbol{\alpha}^\mathsf{T}\mathbf{Y}^\mathsf{T}\mathbf{y}\right) && (13)\\
&= r_0-\boldsymbol{\alpha}^\mathsf{T}\mathbf{r}_1 && (14)
\end{aligned}$$

where $\mathbf{r}_1$ denotes $(r_1, r_2, \ldots, r_P)^\mathsf{T}$.

A.4 MAP solution

All MAP solutions follow from the full joint:

$$p(\mathbf{y},\boldsymbol{\alpha},\beta) = p(\mathbf{y}\mid\boldsymbol{\alpha},\beta)\,p(\boldsymbol{\alpha})\,p(\beta) \tag{15}$$

A.5 Conjugate priors

If we model $p(\boldsymbol{\alpha})$ as a zero mean Gaussian with variance equal to $1/\lambda$ (so λ is a *precision*), and put an inverse gamma prior on β^2, we have

$$\begin{aligned}
p\left(\mathbf{y},\boldsymbol{\alpha},\beta^2\right) = \frac{1}{\sqrt{2\pi\beta^2}^{N-P}}\exp\left(-\frac{1}{2\beta^2}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})\right)&\\
\times\frac{1}{\sqrt{2\pi/\lambda}^{P}}\exp\left(-\frac{\lambda}{2}\boldsymbol{\alpha}^\mathsf{T}\boldsymbol{\alpha}\right)\frac{\delta^\nu}{\Gamma(\nu)}\beta^{-2(\nu+1)}\exp\left(-\frac{\delta}{\beta^2}\right)& \quad (16)
\end{aligned}$$

To estimate $\boldsymbol{\alpha}$,

$$\begin{aligned}
\log p\left(\mathbf{y},\boldsymbol{\alpha},\beta^2\right) &= -\frac{1}{2\beta^2}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})-\frac{\lambda}{2}\boldsymbol{\alpha}^\mathsf{T}\boldsymbol{\alpha}+C && (17)\\
\frac{\partial}{\partial\boldsymbol{\alpha}}\log p\left(\mathbf{y},\boldsymbol{\alpha},\beta^2\right) &= \frac{1}{\beta^2}\mathbf{Y}^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})-\lambda\boldsymbol{\alpha}=0 && (18)\\
\hat{\boldsymbol{\alpha}} &= (\mathbf{Y}^\mathsf{T}\mathbf{Y}+\lambda\beta^2\mathbf{I})^{-1}\mathbf{Y}^\mathsf{T}\mathbf{y}. && (19)
\end{aligned}$$

So, it amounts to just adding $\lambda\beta^2$ to r_0.

Note: Normally, in the normal inverse gamma distribution, the variance of $p(\boldsymbol{\alpha})$ is proportional to β^2. In this case, however, that is not appropriate; they are independent. This leads to the estimate of $\boldsymbol{\alpha}$ depending on β.

Similarly for β^2,

$$\begin{aligned}
\log p\left(\mathbf{y},\boldsymbol{\alpha},\beta^2\right) &= (N-P)\log\sqrt{\beta^2}-\frac{1}{2\beta^2}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})-(\nu+1)\log\beta^2-\frac{\delta}{\beta^2}+C && (20)\\
\frac{\partial}{\partial\beta^2}\log p\left(\mathbf{y},\boldsymbol{\alpha},\beta^2\right) &= \frac{N-P}{2\beta^2}+\frac{1}{2\beta^4}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})-\frac{\nu+1}{\beta^2}+\frac{\delta}{\beta^2}=0 && (21)\\
\hat{\beta}^2 &= \frac{1}{N-P+2(\nu+1)}\left((\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})^\mathsf{T}(\mathbf{y}-\mathbf{Y}\boldsymbol{\alpha})+2\delta\right) && (22)\\
&= \frac{1}{N-P+2(\nu+1)}\left(\mathbf{y}^\mathsf{T}\mathbf{y}+\boldsymbol{\alpha}^\mathsf{T}(\mathbf{Y}^\mathsf{T}\mathbf{Y}\boldsymbol{\alpha}-2\mathbf{Y}^\mathsf{T}\mathbf{y})\right)+\frac{2\delta}{N-P+2(\nu+1)}. && (23)
\end{aligned}$$

Note that the matrix term is just the excitation energy, but can be calculated more quickly using the terms $\mathbf{Y}^\mathsf{T}\mathbf{Y}$ and $\mathbf{Y}^\mathsf{T}\mathbf{y}$ already calculated above.

A.6 Polynomials

The linear model is

$$y_n = \beta x_n + \sum_{p=1}^{P} \alpha_p y_{n-p}. \tag{24}$$

Taking the z-transform,

$$y = \beta x + \sum_{p=1}^{P} \alpha_p y z^{-p} \tag{25}$$

$$y\left(1 - \sum_{p=1}^{P} \alpha_p z^{-p}\right) = \beta x \tag{26}$$

$$\frac{y}{x} = H(z) = \frac{\beta}{1 - \sum_{p=1}^{P} \alpha_p z^{-p}}. \tag{27}$$

The polynomial in the denominator defines the poles; multiplying through by z^P,

$$z^P - \alpha_1 z^{P-1} - \alpha_2 s^{P-2} - \cdots - \alpha_P = 0 \tag{28}$$

$$(z - \rho_1 e^{j\theta_1})(z - \rho_2 e^{j\theta_2}) \dots (z - \rho_P e^{j\theta_P}) = 0. \tag{29}$$

The poles appear as conjugate pairs, with one on the real line for odd orders.

In fact, for resonance matching, the order will be even. For second order:

$$(z - \rho_1 e^{j\theta_1})(z - \rho_1 e^{-j\theta_1}) = 0 \tag{30}$$

$$z^2 - z\rho_1 e^{j\theta_1} - z\rho_1 e^{-j\theta_1} + \rho_1^2 = 0 \tag{31}$$

$$z^2 - 2z\rho_1 \cos\theta_1 + \rho_1^2 = 0. \tag{32}$$

A.7 Recursions

This is based on Atal's method, but I guess the technique is somewhat older. The key is equate the z transforms of the log magnitude spectrum and the cepstrum:

$$\log\left[\frac{\beta}{1 - \sum_{p=1}^{P} \alpha_p z^{-p}}\right] = \sum_{n=0}^{\infty} c_n z^{-n} \tag{33}$$

$$\log\beta - \log\left[1 - \sum_{p=1}^{P} \alpha_p z^{-p}\right] = c_0 + \sum_{n=1}^{\infty} c_n z^{-n}. \tag{34}$$

So take $c_0 = \log\beta$ and differentiate the remaining terms to get rid of the logarithm:

$$-\frac{d}{dz^{-1}} \log\left[1 - \sum_{p=1}^{P} \alpha_p z^{-p}\right] = \frac{d}{dz^{-1}}\left[\sum_{n=1}^{\infty} c_n z^{-n}\right] \tag{35}$$

$$\sum_{p=1}^{P} p\alpha_p z^{-p+1} = \sum_{n=1}^{\infty} n c_n z^{-n+1}\left(1 - \sum_{p=1}^{P} \alpha_p z^{-p}\right). \tag{36}$$

Equating terms in z^{-1} (beginning with the constant again)

$$\alpha_1 = c_1, \tag{37}$$

$$2\alpha_2 = 2c_2 - c_1\alpha_1, \tag{38}$$

$$3\alpha_3 = 3c_3 - 2c_2\alpha_1 - c_1\alpha_2, \tag{39}$$

$$4\alpha_4 = 4c_4 - 3c_3\alpha_1 - 2c_2\alpha_2 - c_1\alpha_3, \tag{40}$$

so, this is initially a recursion to give α_n in terms of cepstra. The general terms appears to be

$$\beta = \exp(c_0) \tag{41}$$

$$\alpha_n = c_n - \sum_{p=1}^{n-1} \frac{p}{n} c_p \alpha_{n-p}. \tag{42}$$

Those equations can be flipped around trivially to give

$$c_1 = \alpha_1, \tag{43}$$

$$2c_2 = 2\alpha_2 + c_1\alpha_1, \tag{44}$$

$$3c_3 = 3\alpha_3 + 2c_2\alpha_1 + c_1\alpha_2, \tag{45}$$

$$4c_4 = 4\alpha_4 + 3c_3\alpha_1 + 2c_2\alpha_2 + c_1\alpha_3, \tag{46}$$

and the general term is

$$c_0 = \log \beta, \tag{47}$$

$$c_n = \alpha_n + \sum_{p=1}^{n-1} \frac{p}{n} c_p \alpha_{n-p}. \tag{48}$$

If we flip the summation and define $p = n - i$ so that $i = n - p$, we get

$$c_n = \alpha_n + \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) c_{n-i}\alpha_i, \tag{49}$$

which is what, e.g., HTK defines.