A Linear Model

A.1 Generative model

This is a general filter:

\[ x_n - \beta_0 \sum y_{n-1} + \sum z^{-1} x_{n-1} - \beta_1 \sum z^{-1} x_{n-2} - \beta_2 \sum z^{-1} x_{n-3} - \beta_3 \sum z^{-1} x_{n-4} \]

\[ y_n = \alpha_1 \sum y_{n-1} + \alpha_2 \sum y_{n-2} + \alpha_3 \sum y_{n-3} \]

All pole is that with zeros removed:

\[ x_n - \beta \sum z^{-1} x_{n-1} - \alpha_1 \sum z^{-1} x_{n-2} - \alpha_2 \sum z^{-1} x_{n-3} - \alpha_3 \sum z^{-1} x_{n-4} \]

\[ y_n = \alpha_1 \sum z^{-1} y_{n-1} + \alpha_2 \sum z^{-1} y_{n-2} + \alpha_3 \sum z^{-1} y_{n-3} \]

Notice that we still have \( \beta_0 \), a gain term, written here as \( \beta \).
A.2 Formulation

Say we have:

- \( N \) acoustic observations, \( y = (y_{t-N+1}, y_{t-N+2}, \ldots, y_t)^T \). They tend to be time indexed, in which case the most recent is time \( t \).
- A vector of \( P \) linear prediction coefficients, \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)^T \).

Define a model where the observation is a function of a current input, or excitation, and previous observations (outputs):

\[
y_n = \beta x_n + \sum_{p=1}^{P} \alpha_p y_{n-p}.
\] (1)

It’s more easily expressed in matrix form. For a window size \( N \):

\[
\begin{pmatrix}
y_{t-N+P+1} \\
y_{t-N+P+2} \\
\vdots \\
y_t
\end{pmatrix}
= \begin{pmatrix}
y_{t-N+1} & y_{t-N+2} & \cdots & y_{t-N+P} \\
y_{t-N+2} & y_{t-N+3} & \cdots & y_{t-N+P+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_{t-P} & y_{t-P+1} & \cdots & y_{t-1}
\end{pmatrix}
\begin{pmatrix}
\alpha_P \\
\alpha_{P-1} \\
\vdots \\
\alpha_1
\end{pmatrix}
+ \beta x
\] (2)

Notice that there are \( N - P \) equations; we ignore the ones for which the previous outputs are not available. Ideally, \( N \gg P \), so there are lots of samples.

A.3 Estimation of parameters

Say the excitation is a Gaussian with zero mean and unit variance:

\[
p(x) = \frac{1}{\sqrt{2\pi N-P}} \exp \left(-\frac{1}{2} x^T x \right) .
\] (3)

Make a change of variable

\[
y = Y \alpha + \beta x
\] (4)

\[
x = \frac{1}{\beta} (y - Y \alpha)
\] (5)

The Jacobian is \( 1/\beta \) for each of the \( N - P \) equations, so after substitution, we get

\[
p(y | \alpha) = \frac{1}{\sqrt{2\pi \beta^2 N-P}} \exp \left(-\frac{1}{2 \beta^2} (y - Y \alpha)^T (y - Y \alpha) \right) .
\] (6)

Finally, differentiate w.r.t. \( \alpha \) and equate to zero:\n
\[
\hat{\alpha} = (Y^T Y)^{-1} Y^T y.
\] (7)

This is not specific to LPC; it’s a standard statistical result.

This bit is specific to LPC:

- \( Y^T Y \) is basically the autocorrelation, with a few edge effects.

- \( Y^T y \) is also basically the autocorrelation, with a few edge effects.

\[
\hat{\alpha} = \begin{pmatrix}
r_0 & r_1 & \cdots & r_{P-1} \\
r_1 & r_0 & \cdots & r_{P-2} \\
\vdots & \vdots & \ddots & \vdots \\
r_{P-1} & r_{P-2} & \cdots & r_0
\end{pmatrix}^{-1}
\begin{pmatrix}
r_P \\
r_{P-1} \\
\vdots \\
r_1
\end{pmatrix}
\] (8)

\[1\text{You can sort of see the result is going to be a rearrangement of } y = Y \alpha.\]
There is a $1/(N-P)$ term that cancels in the above; $Y^Ty$ and $Y^Ty$ are not normalised, but the autocorrelation terms are.

The gain is just the variance:

$$\hat{\beta}^2 = \frac{1}{N-P}(y - Y\alpha)^T(y - Y\alpha)$$  \hspace{1cm} (9)

$$= \frac{1}{N-P}(y - Y(Y^TY)^{-1}Y^Ty)^T(y - Y(Y^TY)^{-1}Y^Ty)$$  \hspace{1cm} (10)

$$= \frac{1}{N-P} \left( y^TY + y^TY(Y^TY)^{-1}Y^TYy - 2y^TY(Y^TY)^{-1}Y^Ty \right)$$  \hspace{1cm} (11)

$$= \frac{1}{N-P} \left( y^TY - y^TY(Y^TY)^{-1}Y^Tyy \right)$$  \hspace{1cm} (12)

$$= \frac{1}{N-P} \left( y^TY - \alpha^TY^Ty \right)$$  \hspace{1cm} (13)

$$= r_0 - \alpha^Tr_1$$  \hspace{1cm} (14)

where $r_1$ denotes $(r_1, r_2, \ldots, r_P)^T$.

### A.4 MAP solution

All MAP solutions follow from the full joint:

$$p(y, \alpha, \beta) = p(y | \alpha, \beta) p(\alpha) p(\beta)$$  \hspace{1cm} (15)

### A.5 Conjugate priors

If we model $p(\alpha)$ as a zero mean Gaussian with variance equal to $1/\lambda$ (so $\lambda$ is a precision), and put an inverse gamma prior on $\beta^2$, we have

$$p(y, \alpha, \beta^2) = \frac{1}{\sqrt{2\pi\beta^2}^{N-P}} \exp \left( -\frac{1}{2\beta^2}(y - Y\alpha)^T(y - Y\alpha) \right) \times \frac{1}{\sqrt{2\pi/\lambda}} \exp \left( -\frac{\lambda}{2} \alpha^T \alpha \right) \frac{\delta^\nu}{\Gamma(\nu)} \beta^{-2(\nu+1)} \exp \left( -\frac{\delta}{\beta^2} \right)$$  \hspace{1cm} (16)

To estimate $\alpha$,

$$\log p(y, \alpha, \beta^2) = -\frac{1}{2\beta^2}(y - Y\alpha)^T(y - Y\alpha) - \frac{\lambda}{2} \alpha^T \alpha + C$$  \hspace{1cm} (17)

$$\frac{\partial}{\partial \alpha} \log p(y, \alpha, \beta^2) = \frac{1}{\beta^2} Y^T(y - Y\alpha) - \lambda \alpha = 0$$  \hspace{1cm} (18)

$$\hat{\alpha} = (Y^TY + \lambda \beta^2 I)^{-1}Y^Ty.$$  \hspace{1cm} (19)

So, it amounts to just adding $\lambda \beta^2$ to $r_0$.

Note: Normally, in the normal inverse gamma distribution, the variance of $p(\alpha)$ is proportional to $\beta^2$. In this case, however, that is not appropriate; they are independent. This leads to the estimate of $\alpha$ depending on $\beta$.

Similarly for $\beta^2$,

$$\log p(y, \alpha, \beta^2) = (N-P) \log \sqrt{\beta^2} - \frac{1}{2\beta^2}(y - Y\alpha)^T(y - Y\alpha) - (\nu + 1) \log \beta^2 - \frac{\delta}{\beta^2} + C$$  \hspace{1cm} (20)

$$\frac{\partial}{\partial \beta^2} \log p(y, \alpha, \beta^2) = \frac{N-P}{2\beta^4} + \frac{1}{2\beta^4}(y - Y\alpha)^T(y - Y\alpha) - \frac{\nu + 1}{\beta^2} + \frac{\delta}{\beta^2} = 0$$  \hspace{1cm} (21)

$$\hat{\beta}^2 = \frac{1}{N-P + 2(\nu + 1)} (y^TY + \alpha^T(Y^TY\alpha - 2Y^Ty))$$  \hspace{1cm} (22)

$$\hat{\beta}^2 = \frac{1}{N-P + 2(\nu + 1)} \left( y^TY + \alpha^T(Y^TY\alpha - 2Y^Ty) \right) + \frac{2\delta}{N-P + 2(\nu + 1)}.$$  \hspace{1cm} (23)

Note that the matrix term is just the excitation energy, but can be calculated more quickly using the terms $Y^TY$ and $Y^Ty$ already calculated above.
A.6 Polynomials

The linear model is

\[ y_n = \beta x_n + \sum_{p=1}^{P} \alpha_p y_{n-p}. \]  

(24)

Taking the z-transform,

\[ y = \beta x + \sum_{p=1}^{P} \alpha_p y z^{-p} \]  

(25)

\[ y \left( 1 - \sum_{p=1}^{P} \alpha_p z^{-p} \right) = \beta x \]  

(26)

\[ \frac{y}{x} = H(z) = \frac{\beta}{1 - \sum_{p=1}^{P} \alpha_p z^{-p}}. \]  

(27)

The polynomial in the denominator defines the poles; multiplying through by \( z^P \),

\[ z^P - \alpha_1 z^{P-1} - \alpha_2 z^{P-2} - \cdots - \alpha_P = 0 \]  

(28)

\[ (z - \rho_1 e^{i\theta_1})(z - \rho_2 e^{i\theta_2}) \cdots (z - \rho_P e^{i\theta_P}) = 0. \]  

(29)

The poles appear as conjugate pairs, with one on the real line for odd orders. In fact, for resonance matching, the order will be even. For second order:

\[ (z - \rho_1 e^{i\theta_1})(z - \rho_1 e^{-i\theta_1}) = 0 \]  

(30)

\[ z^2 - z\rho_1 e^{i\theta_1} + \rho_1^2 = 0 \]  

(31)

\[ z^2 - 2z\rho_1 \cos \theta_1 + \rho_1^2 = 0. \]  

(32)

A.7 Recursions

This is based on Atal’s method, but I guess the technique is somewhat older. The key is equate the \( z \) transforms of the log magnitude spectrum and the cepstrum:

\[ \log \left[ \frac{\beta}{1 - \sum_{p=1}^{P} \alpha_p z^{-p}} \right] = \sum_{n=0}^{\infty} c_n z^{-n} \]  

(33)

\[ \log \beta - \log \left[ 1 - \sum_{p=1}^{P} \alpha_p z^{-p} \right] = c_0 + \sum_{n=1}^{\infty} c_n z^{-n}. \]  

(34)

So take \( c_0 = \log \beta \) and differentiate the remaining terms to get rid of the logarithm:

\[ -\frac{d}{dz^{-1}} \log \left[ 1 - \sum_{p=1}^{P} \alpha_p z^{-p} \right] = \frac{d}{dz^{-1}} \left[ \sum_{n=1}^{\infty} c_n z^{-n} \right] \]  

(35)

\[ \sum_{p=1}^{P} p\alpha_p z^{-p+1} = \sum_{n=1}^{\infty} n c_n z^{-n+1} \left( 1 - \sum_{p=1}^{P} \alpha_p z^{-p} \right). \]  

(36)

Equating terms in \( z^{-1} \) (beginning with the constant again)

\[ \alpha_1 = c_1, \]  

(37)

\[ 2\alpha_2 = 2c_2 - c_1 \alpha_1, \]  

(38)

\[ 3\alpha_3 = 3c_3 - 2c_2 \alpha_1 - c_1 \alpha_2, \]  

(39)

\[ 4\alpha_4 = 4c_4 - 3c_3 \alpha_1 - 2c_2 \alpha_2 - c_1 \alpha_3, \]  

(40)
so, this is initially a recursion to give $\alpha_n$ in terms of cepstra. The general terms appears to be

$$\beta = \exp(c_0)$$

$$\alpha_n = c_n - \sum_{p=1}^{n-1} \frac{p}{n} c_p \alpha_{n-p}. \tag{42}$$

Those equations can be flipped around trivially to give

$$c_1 = \alpha_1, \tag{43}$$
$$2c_2 = 2\alpha_2 + c_1 \alpha_1, \tag{44}$$
$$3c_3 = 3\alpha_3 + 2c_2 \alpha_1 + c_1 \alpha_2, \tag{45}$$
$$4c_4 = 4\alpha_4 + 3c_3 \alpha_1 + 2c_2 \alpha_2 + c_1 \alpha_3, \tag{46}$$

and the general term is

$$c_0 = \log \beta,$$

$$c_n = \alpha_n + \sum_{p=1}^{n-1} \frac{p}{n} c_p \alpha_{n-p}. \tag{48}$$

If we flip the summation and define $p = n - i$ so that $i = n - p$, we get

$$c_n = \alpha_n + \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) c_{n-i} \alpha_i, \tag{49}$$

which is what, e.g., HTK defines.